Lemma:
Let \( G = (V, E) \) be an undirected graph, and let \( f : V \times V \rightarrow (\mathbb{R}^+ \cup +\infty) \) be a cost (a.k.a. weight) function for \( G \).
Let \( F = \{ (V_1, E_1), (V_2, E_2), \ldots, (V_k, E_k) \} \) be a minimal spanning forest for \( G \). Let
\[
\hat{E} = \bigcup_{i=1}^{k} E_i
\]
be the set of all edges in the spanning forest \( F \).
Let \( e = (u, v) \) be the edge of lowest cost chosen by Kruskal’s algorithm. Then, there exists a spanning tree \( T \) which includes \( \hat{E} \cup \{e\} \) which is as low cost as any spanning tree which includes \( \hat{E} \).

Proof: (by contradiction)
Suppose there exists a spanning tree \( T' = (V, E') \) which includes \( \hat{E} \) but not \( e \), and \( \text{cost}(T') \) is (strictly) less than the cost of any spanning tree that includes \( \hat{E} \cup \{e\} \).

We reach a contradiction by constructing a new spanning tree \( T'' = (V, E'') \) that includes \( \hat{E} \) and:
\[
\text{cost}(T'') \leq \text{cost}(T')
\]

Construction of \( T'' \):
Recall that the edge \( e \) chosen by Kruskal’s algorithm connects vertices \( u \) and \( v \); i.e., \( e = (u, v) \). The vertices \( u \) and \( v \) must be connected in \( T' \) (because \( T' \) is a tree). The tree \( T' \) does not include \( e \), therefore if we add \( e \) to \( T' \), a cycle results. Let \( e' \) denote an edge on that cycle such that \( e' \neq e \) and \( e' \notin \hat{E} \). We know such an edge exists because \( u \) and \( v \) are connected in \( T' \), but \( u \) and \( v \) are not connected in \( F \).
Let \( T'' = (V, E'') \) where
\[
E'' = E' - \{e'\} \cup \{e\}
\]
The conditions under which \( e \) is chosen by Kruskal’s algorithm guarantee that
\[
f(e) \leq f(e')
\]
Therefore:
\[
\text{cost}(T'') \leq \text{cost}(T') \quad (1)
\]
Inequality (1) is a contradiction, because the proof hypothesis requires:
\[
\text{cost}(T'') > \text{cost}(T')
\]