CSC721 **Spring 2013**

Algorithms

Lemma:

Let $G = (V, E)$ be an undirected graph, and let $f : V \times V \rightarrow (\mathcal{R}^+ \cup +\infty)$ be a cost (a.k.a. weight) function for G.

Let $F = \{ \ (V_1, E_1), (V_2, E_2), ..., (V_k, E_k) \ \}$ be a minimal spanning forest for G. Let

$$\hat{E} = \cup_{i=1}^{k} \ E_i$$

be the set of all edges in the spanning forest F.

Let $e = (u, v)$ be the edge of lowest cost chosen by Kruskal's algorithm. Then, there exists a spanning tree T which includes $\hat{E} \cup \{e\}$ which is as low cost as any spanning tree which includes $\hat{E}$.

Proof: (by contradiction)

Suppose there exists a spanning tree $T' = (V, E')$ which includes $\hat{E}$ but not e, and cost(T') is (strictly) less than the cost of any spanning tree that includes $\hat{E} \cup \{e\}$.

We reach a contradiction by constructing a new spanning tree $T'' = (V, E'')$ that includes $\hat{E}$ and:

$$\text{cost}(T'') \leq \text{cost}(T')$$

Construction of T'' :

Recall that the edge e chosen by Kruskal's algorithm connects vertices u and v; i.e., $e = (u, v)$. The vertices u and v must be connected in T' (because T' is a tree). The tree T' does not include e, therefore if we add e to T', a cycle results. Let e' denote an edge on that cycle such that $e' \neq e$ and $e' \notin \hat{E}$. We know such an edge exists because u and v are connected in T', but u and v are not connected in F.

Let $T'' = (V, E'')$ where

$$E'' = E' - \{e'\} \cup \{e\} \ \ .$$

The conditions under which e is chosen by Kruskal's algorithm guarantee that

$$f(e) \leq f(e')$$

Therefore:

$$\text{cost}(T'') \leq \text{cost}(T') \qquad (1)$$

Inequality (1) is a contradiction, because the proof hypothesis requires:

$$\text{cost}(T'') > \text{cost}(T')$$