

Theorem: The vertex cover problem is \mathcal{NP} -complete.

Definition: Let $G = (V, E)$ be an undirected graph.

- A vertex cover of G is a subset $S \subseteq V$ such that every edge in E is incident on some vertex in S .
- The vertex cover problem can be stated as follows: Given an undirected graph $G = (V, E)$ and a positive integer k , where $1 \leq k \leq |V|$, does G have a vertex cover of size k ?

Theorem: The vertex cover problem is \mathcal{NP} -complete.

Proof: The key idea of the proof is to construct a polynomial time mapping reduction from the clique problem to the vertex cover problem.

Let $G = (V, E)$ denote an undirected graph and let k be a positive integer with $1 \leq k \leq |V|$. Think of G and k as an instance of the clique problem.

We construct a new graph \overline{G} , such that G has a k -clique if and only if \overline{G} has a vertex cover size $(|V| - k)$.

Define $\overline{G} = (V, \overline{E})$ where $\overline{E} = V \times V - E$. I.e., \overline{G} has the same vertex set as G , and \overline{G} contains all of the edges that are **not** in G . It is clear that \overline{G} can be constructed from G in polynomial time.

We now turn our attention to verifying the claim G has a k clique if and only if \overline{G} has a vertex cover size $(|V| - k)$.

“ \Rightarrow ” Suppose G has a k -clique. Let $S \subseteq V$ denote the vertices in the k -clique. We claim that the set of vertices $V - S$ forms a vertex cover in \overline{G} . To establish this fact, we must show that every edge (v, w) in \overline{E} is incident on some vertex in $V - S$.

Consider an arbitrary edge (v, w) in \overline{E} . It is not possible for both vertices v and w to be in S . If both v and w were in S then the edge (v, w) would be in E (instead of \overline{E}), since *every* pair of vertices in S is connected by an edge in E . At least one of v or w must be in $V - S$ and therefore the edge (v, w) is incident on some vertex in $V - S$. The edge (v, w) is an arbitrary edge in \overline{E} , therefore we can conclude that **every** edge in \overline{E} is incident on some vertex in $V - S$. Therefore, $V - S$ is a vertex cover of \overline{G} .

“ \Leftarrow ” Suppose \overline{G} has a vertex cover size $(|V| - k)$. Let $T \subseteq V$ denote the set of vertices comprising the vertex cover of \overline{G} .

Consider any pair of vertices x and y in $V - T$. Recall that the edge sets E and \overline{E} are complementary in $V \times V$, thus the edge (x, y) must be either in E or \overline{E} . However, the edge (x, y) can not belong to \overline{E} because every edge in \overline{E} is incident on a vertex in T , but both x and y are in $V - T$. Thus, every pair of vertices in $V - T$ is connected by an edge in E . By definition of a clique, and the fact that $|V - T| = k$ we can conclude that the vertex set $V - T$ forms a k -clique in G .