CSC721 Algorithms   Matrix Operations   Fall 2017

Recall our discussion of inverting an upper-triangular non-singular $n \times n$ matrix A. The basic idea is to use a divide and conquer approach. For the moment, assume that $n = 2^k$ for some $k \geq 0$. We partition the matrix as follows:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}$$

We have a simple formula for the inverse of A as follows:

$$A^{-1} = \begin{bmatrix} A_{1,1}^{-1} & -A_{1,1}^{-1} A_{1,2} A_{2,2}^{-1} \\ 0 & A_{2,2}^{-1} \end{bmatrix} \tag{1}$$

Let $M(n)$ denote the time needed to multiply two $n \times n$ matrices. Further suppose that $M(n)$ satisfies:

$$4M(n) \leq M(2n) \leq 8M(n) \ . \tag{2}$$

Equation (2) is equivalent to requiring that $M(n)$ satisfy:

$$M(n) \in \Omega(n^2) \quad \text{and} \quad M(n) \in \mathcal{O}(n^3) \ .$$

If we apply equation (1) recursively, we have a divide and conquer algorithm with asymptotic complexity $T(n)$, where:

$$\begin{aligned} T(1) &= c \\ T(n) &= 2T(n/2) + 2M(n/2) + (n/2)^2 \end{aligned} \tag{3}$$

Observe the terms on the right hand side of equation (3) in relation to equation (1). The terms correspond to:

1. The term $2T(n/2)$ gives the time needed to solve the two sub-problems $A_{1,1}^{-1}$ and $A_{2,2}^{-1}$. Notice these sub-problems are size $n/2$.
2. The term $2M(n/2)$ gives the time needed to perform the two multiplications involved in the upper right block of A^{-1}.
3. The term $(n/2)^2$ gives the time needed to change the sign of each element (i.e., apply the unary minus) in the upper right block of A^{-1}.

From equation (2), we may conclude that

$$(n/2)^2 \leq M(n/2) \ . \tag{4}$$

Substituting equation (2) into (3) we have:

$$\begin{aligned} T(1) &= c \\ T(n) &\leq 2T(n/2) + 3M(n/2) \end{aligned} \tag{5}$$

We solve the recurrence equation by writing out the usual table of equations.

$$\begin{aligned}
T(n) &\leq 2T(n/2) + 3M(n/2) \\
T(n/2) &\leq 2T(n/4) + 3M(n/4) \\
T(n/4) &\leq 2T(n/8) + 3M(n/8) \\
&\vdots \\
T(n/2^{k-1}) &\leq 2T(1) + 3M(n/2^k)
\end{aligned}$$

We need multiply each equaiton above by the appropriate power of two so that cancellation will occur when we sum.

$$\begin{array}{cccl}
 & T(n) & \leq & 2T(n/2) + 3M(n/2) \\
+ & 2T(n/2) & \leq & 2^2T(n/4) + 3M(n/4) \cdot 2 \\
+ & 2^2T(n/4) & \leq & 2^3T(n/8) + 3M(n/8) \cdot 2^2 \\
+ & . & & \\
+ & . & & \\
+ & . & & \\
+ & 2^{k-1}T(n/2^{k-1}) & \leq & 2^kT(1) + 3M(n/2^k) \cdot 2^{k-1} \\
 & --------- & & \\
 & T(n) & \leq & cn + 3\left[M(n/2) + 2M(n/4) + 2^2M(n/8) + ... + 2^{k-1}M(n/2^k)\right] \\
 & & = & cn + \frac{3}{2}\left[2M(n/2) + 2^2M(n/4) + 2^3M(n/8) + ... + 2^kM(n/2^k)\right]
\end{array} \tag{6}$$

The sum in the last expression above can be simplified using the first part of our hypothesis regarding $M(n)$. Recall:

$$4M(n) \leq M(2n) \tag{7}$$

We can re-formulate inequality (7) as:

$$M(n/2) \leq \frac{1}{4}M(n) \tag{8}$$

We can also generalize inequality (8) to j factors of 2 in the denominator.

$$M(n/2^j) \leq \frac{1}{4^j}M(n) \tag{9}$$

We apply inequality (9) to each term in the last result from equation (6) to obtain:

$$\begin{aligned}
T(n) &\leq cn + \frac{3}{2}\left[\frac{2}{4}M(n) + \frac{2^2}{4^2}M(n) + \frac{2^3}{4^3}M(n) + ... + \frac{2^k}{4^k}M(n)\right] \\
&= cn + \frac{3}{2}M(n)\left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^k}\right] \\
&\in \mathcal{O}\left(M(n)\right)
\end{aligned} \tag{10}$$