CSC721 Algorithms Fall 2017

Matrix Operations

Matrix Inverse:

Let A be an $n \times n$ matrix where $n = 2^k$ for some $k > 0$. Let A be partitioned as:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \tag{1}$$

and suppose $A_{1,1}^{-1}$ exists. We define:

$$\Delta = A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2} \tag{2}$$

and suppose Δ^{-1} exists. Then:

$$A^{-1} = \begin{bmatrix} A_{1,1}^{-1} + A_{1,1}^{-1}A_{1,2}\Delta^{-1}A_{2,1}A_{1,1}^{-1} & -A_{1,1}^{-1}A_{1,2}\Delta^{-1} \\ -\Delta^{-1}A_{2,1}A_{1,1}^{-1} & \Delta^{-1} \end{bmatrix} \tag{3}$$

Proof:

By simple algebra we can verify that:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{2,1}A_{1,1}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{1,1} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} I & A_{1,1}^{-1}A_{1,2} \\ 0 & I \end{bmatrix} \tag{4}$$

where Δ is given in equation (2).

Recall that the inverse of a matrix product is the product of the matrix inverses in reverse order. Fortunately, the three factors on the right side of equation (4) are easy to invert. Therefore,

$$\begin{aligned} A^{-1} &= \begin{bmatrix} I & -A_{1,1}^{-1}A_{1,2} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{1,1}^{-1} & 0 \\ 0 & \Delta^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{2,1}A_{1,1}^{-1} & I \end{bmatrix} \\ &= \begin{bmatrix} A_{1,1}^{-1} + A_{1,1}^{-1}A_{1,2}\Delta^{-1}A_{2,1}A_{1,1}^{-1} & -A_{1,1}^{-1}A_{1,2}\Delta^{-1} \\ -\Delta^{-1}A_{2,1}A_{1,1}^{-1} & \Delta^{-1} \end{bmatrix} \end{aligned} \tag{5}$$

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In the special case where A is a non-singular upper triangular[1] matrix (i.e., $A_{2,1} = 0$) then equation (5) simplifies to:

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}^{-1} = \begin{bmatrix} A_{1,1}^{-1} & -A_{1,1}^{-1}A_{1,2}A_{2,2}^{-1} \\ 0 & A_{2,2}^{-1} \end{bmatrix} \tag{6}$$

[1] A similar result holds for non-singular lower triangular matrices.