Optimal Order for Matrix Multiplication

Algorithm

Input: Matrix sizes $r_0, r_1, ..., r_n$ corresponding to a matrix product $\hat{M} = M_1 \times M_2 \times ... \times M_n$

Output: Optimal Number of scalar operations (multiplies) for the matrix product.
Method:

1. **for** $i = 1$ to n do $\quad m_{i,i} = 0 \quad ;$
2. **for** $\ell = 1$ to $(n-1)$ do
3.   **for** $i = 1$ to $(n - \ell)$ do
4. $j = \ell + i \quad ;$
5. $m_{i,j} \; = \; \min_{i \le k < j} \;\; (m_{i,k} \, + \, m_{k+1,j} \, + \, r_{i-1} r_k r_j)$
6.   **end for** i
7. **end for** ℓ

Analysis

Analysis proceeds by a simple count of iterations, working from the innermost loop outward. The additions and multiplications on line 5 take a constant amount of time. To find the minimum requires:

$$T_{\text{inner}} \; = \; c(j-i) \; = \; c\ell$$

We then consider the middle loop.

$$\begin{aligned} T_{\text{middle}} &= \textstyle\sum_{i=1}^{n-\ell} \; T_{\text{inner}} \\ &= \textstyle\sum_{i=1}^{n-\ell} \; c\ell \\ &= c\ell(n-\ell) \\ &= c(n\ell - \ell^2) \end{aligned}$$

Finally, we consider the outer loop.

$$\begin{aligned} T_{\text{outer}}(n) &= \textstyle\sum_{\ell=1}^{n-1} \; T_{\text{middle}} \\ &= \textstyle\sum_{\ell=1}^{n-1} \; c(n\ell - \ell^2) \\ &= cn \textstyle\sum_{\ell=1}^{n-1} \; \ell \quad - \quad \sum_{\ell=1}^{n-1} \; \ell^2 \\ &= c \; n \, \tfrac{(n-1)\, n}{2} \quad - \quad c \, \tfrac{(n-1)n(2n-1)}{6} \\ &= c \left(\left(\tfrac{n^3}{2} - \tfrac{n^2}{2} \right) \; - \; \left(\tfrac{n^3}{3} - \tfrac{n^2}{2} + \tfrac{n}{6} \right) \right) \\ &= c \left(\tfrac{n^3 + n}{6} \right) \\ &\in \mathcal{O}(n^3) \end{aligned}$$