Definition: Let $G = (V, E)$ be an undirected graph.

- A **vertex cover** of $G$ is a subset $S \subseteq V$ such that every edge in $E$ is incident on some vertex in $S$.
- The vertex cover problem can be stated as follows: Given an undirected graph $G = (V, E)$ and a positive integer $k$, where $1 \leq k \leq |V|$, does $G$ have a vertex cover of size $k$?

**Theorem:** The vertex cover problem is $\mathcal{NP}$-complete.

**Proof:** The key idea of the proof is to construct a polynomial time mapping reduction from the clique problem to the vertex cover problem.

Let $G = (V, E)$ denote an undirected graph and let $k$ be a positive integer with $1 \leq k \leq |V|$. Think of $G$ and $k$ as an instance of the clique problem.

We construct a new graph $\overline{G}$, such that $G$ has a $k$-clique if and only if $\overline{G}$ has a vertex cover size $(|V| - k)$.

Define $\overline{G} = (V, \overline{E})$ where $\overline{E} = V \times V - E$. I.e., $\overline{G}$ has the same vertex set as $G$, and $\overline{G}$ contains all of the edges that are not in $G$. It is clear that $\overline{G}$ can be constructed from $G$ in polynomial time.

We now turn our attention to verifying the claim $G$ has a $k$ clique if and only if $\overline{G}$ has a vertex cover size $(|V| - k)$.

$\Rightarrow$ Suppose $G$ has a $k$-clique Let $S \subseteq V$ denote the vertices in the $k$-clique. We claim that the set of vertices $V - S$ forms a vertex cover in $\overline{G}$. To establish this fact, we must show that every edge $(v, w)$ in $\overline{E}$ is incident on some vertex in $V - S$.

Consider an arbitrary edge $(v, w)$ in $\overline{E}$. It is not possible for both vertices $v$ and $w$ to be in $S$. If both $v$ and $w$ were in $S$ then the edge $(v, w)$ would be in $E$ (instead of $\overline{E}$), since every pair of vertices in $S$ is connected by an edge in $E$. At least one of $v$ or $w$ must be in $V - S$ and therefore the edge $(v, w)$ is incident on some vertex in $V - S$. The edge $(v, w)$ is an arbitrary edge in $\overline{E}$, therefore we can conclude that every edge in $\overline{E}$ is incident on some vertex in $V - S$. Therefore, $V - S$ is a vertex cover of $\overline{G}$.

$\Leftarrow$ Suppose $\overline{G}$ has a vertex cover size $(|V| - k)$. Let $T \subseteq V$ denote the set of vertices comprising the vertex cover of $\overline{G}$.

Consider any pair of vertices in $x$ and $y$ in $V - T$. Recall that the edge sets $E$ and $\overline{E}$ are complementary in $V \times V$, thus the edge $(x, y)$ must be either in $E$ or $\overline{E}$. However, the edge $(x, y)$ can not belong to $\overline{E}$ because every edge in $\overline{E}$ is incident on a vertex in $T$, but both $x$ and $y$ are in $V - T$. Thus, every pair of vertices in $V - T$ is connected by an edge in $E$. By definition of a clique, and the fact that $|V - T| = k$ we can conclude that the vertex set $V - T$ forms a $k$-clique in $G$. 