CSC 222 Algorithms Fall 2014

Theorem: The vertex cover problem is $\mathcal{NP}$-complete.

Definition: Let $G = (V, E)$ be an undirected graph.

- A vertex cover of G is a subset $S \subseteq V$ such that every edge in E is incident on some vertex in S.
- The vertex cover problem can be stated as follows: Given an undirected graph $G = (V, E)$ and a positive integer k, where $1 \leq k \leq |V|$, does G have a vertex cover of size k ?

Theorem: The vertex cover problem is $\mathcal{NP}$-complete.

Proof: The key idea of the proof is to construct a polynomial time mapping reduction from the clique problem to the vertex cover problem.

Let $G = (V, E)$ denote an undirected graph and let k be a positive integer with $1 \leq k \leq |V|$. Think of G and k as an instance of the clique problem.

We construct a new graph $\overline{G}$, such that G has a k-clique if and only if $\overline{G}$ has a vertex cover size $(\,|V| - k\,)$.

Define $\overline{G} = (V, \overline{E})$ where $\overline{E} = V \times V \,-\, E$. I.e., $\overline{G}$ has the same vertex set as G, and $\overline{G}$ contains all of the edges that are **not** in G. It is clear that $\overline{G}$ can be constructed from G in polynomial time.

We now turn our attention to verifying the claim G has a k clique if and only if $\overline{G}$ has a vertex cover size $(\,|V| - k\,)$.

"$\Rightarrow$" Suppose G has a k-clique Let $S \subseteq V$ denote the vertices in the k-clique. We claim that the set of vertices $V - S$ forms a vertex cover in $\overline{G}$. To establish this fact, we must show that every edge (v, w) in $\overline{E}$ is incident on some vertex in $V - S$.

Consider an arbitrary edge (v, w) in $\overline{E}$. It is not possible for both vertices v and w to be in S. If both v and w were in S then the edge (v, w) would be in E (instead of $\overline{E}$), since *every* pair of vertices in S is connected by an edge in E. At least one of v or w must be in $V - S$ and therefore the edge (v, w) is incident on some vertex in $V - S$. The edge (v, w) is an arbitrary edge in $\overline{E}$, therefore we can conclude that **every** edge in $\overline{E}$ is incident on some vertex in $V - S$. Therefore, $V - S$ is a vertex cover of $\overline{G}$.

"$\Leftarrow$" Suppose $\overline{G}$ has a vertex cover size $(\,|V| - k\,)$. Let $T \subseteq V$ denote the set of vertices comprising the vertex cover of $\overline{G}$.

Consider any pair of vertices in x and y in $V - T$. Recall that the edge sets E and $\overline{E}$ are complementary in $V \times V$, thus the edge (x, y) must be either in E or $\overline{E}$. However, the edge (x, y) can not belong to $\overline{E}$ because every edge in $\overline{E}$ is incident on a vertex in T, but both x and y are in $V - T$. Thus, every pair of vertices in $V - T$ is connected by an edge in E. By definition of a clique, and the fact that $|V - T| = k$ we can conclude that the vertex set $V - T$ forms a k-clique in G.