Some \(\mathcal{NP}\)-complete problems

**Definitions:** Let \(G = (V, E)\) be an undirected graph.

- A **vertex cover** of \(G\) is a subset \(S \subseteq V\) such that each edge in \(G\) is incident on some vertex in \(S\).
- A **Hamiltonian circuit** of \(G\) is a cycle in \(G\) that contains every vertex of \(V\).
- A graph \(G\) is **\(k\)-colorable** if there exists an assignment of integers \(\{1, 2, 3, \ldots, k\}\), i.e., the “colors”, to the vertices of \(G\) such that no two adjacent vertices have the same color.
- The **chromatic number** of a graph \(G\) is the smallest integer \(k\) such that \(G\) is \(k\)-colorable.
- Given a collection of sets \(S_1, S_2, \ldots, S_n\), does there exist a sub-collection of \(k\) sets \(S_{j_1}, S_{j_2}, S_{j_3}, \ldots, S_{j_k}\) such that
  \[
  \bigcup_{i=1}^{n} S_i = \bigcup_{j=1}^{k} S_{j_i}
  \]

**More Definitions:**
Let \(G = (V, E)\) be a directed graph.

- A **feedback vertex set** of \(G\) is a subset \(S \subseteq V\) such that every cycle of \(G\) contains a vertex in \(S\).
- A **feedback edge set** of \(G\) is a subset \(F \subseteq E\) such that every cycle of \(G\) contains an edge in \(F\).
- A **directed Hamiltonian circuit** of \(G\) is a cycle in \(G\) that contains every vertex of \(V\).

**Some \(\mathcal{NP}\)-complete problems**

- Is a boolean expression satisfiable?
- Does an undirected graph \(G\) have a clique of size \(k\)?
- Does an undirected graph \(G\) have a vertex cover of size \(k\)?
- Does an undirected graph \(G\) have a Hamiltonian circuit?
- Is an undirected graph \(G\) \(k\)-colorable?
- Does a directed graph \(G\) have a feedback vertex set of size \(k\)?
- Does a directed graph \(G\) have a feedback edge set of size \(k\)?
- Does a directed graph \(G\) have a directed Hamiltonian circuit?
- Does a collection of sets \(S_1, S_2, \ldots, S_n\) have a set cover of size \(k\)?
- Knapsack: Given a set of integers \(T = \{t_1, t_2, \ldots, t_n\}\), and an integer \(K\), does there exist a subset \(S \subseteq T\) such that
  \[
  \sum_{x \in S} x = K
  \]