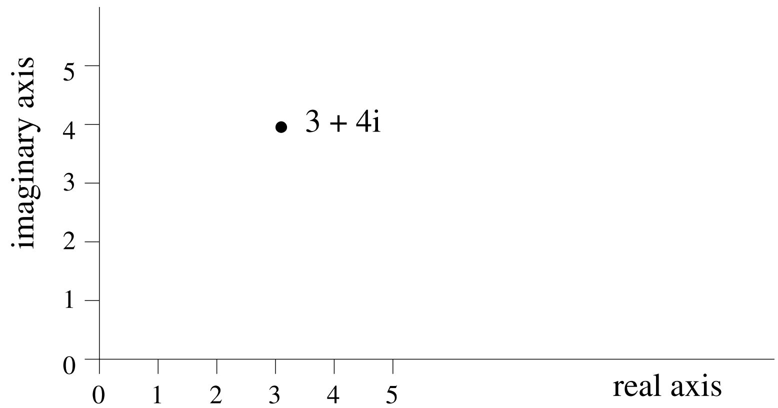
CSC111 Lab 5 Supplement Fall 2017

If x is a real number, we all are familiar with the fact that $x^2 \geq 0$. Not content with real numbers, mathematicians have discovered complex numbers. Complex numbers are based on the imaginary unit:

$$\imath \quad \text{such that } \imath^2 = -1$$

You will see some textbooks write $\imath = \sqrt{-1}$.

Complex numbers can be thought of as a point in the complex plane, and written as: $a+b\imath$. The point $3 + 4\imath$ is illustrated below:



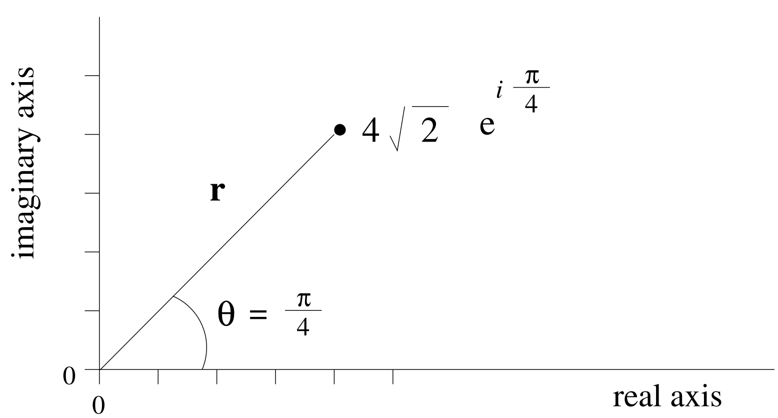
The **absolute value** of a complex number z (denoted by $|z|$) is the distance from the origin. I.e., if $z = a + b\imath$, then

$$|z| = \sqrt{a^2 + b^2} \; .$$

The **conjugate** of a complex number z (denoted by $\overline{z}$) is the reflection about the real axis. I.e., if $z = a + b\imath$, then

$$\overline{z} = a - b\imath \; .$$

Complex numbers can be represented in polar coordinates in the form $re^{\imath\theta}$ where r is the distance from the origin, and θ is the angle relative the the real axis.[1] The point $4 + 4\imath$ is illustrated below in polar coordinates. below:



[1] The polar form for complex numbers is based on the formula $e^{\imath\theta} = \cos(\theta) + \imath\sin(\theta)$. This formula is justified by Taylor's Theorem from the Calculus.

Addition of Complex Numbers Given two complex numbers $z_1 = a+b\imath$ and $z_2 = c+d\imath$, we define addition as

$$z_1 \, + \, z_2 \; = \; (a+c) \, + (b+d)\imath$$

Subtraction is similar.

Multiplication of Complex Numbers Given two complex numbers $z_1 = a + b\imath$ and $z_2 = c + d\imath$, we define multiplication as:

$$z_1 \, z_2 \; = \; (ac - bd) \, + (ad + bc)\imath$$

Multiplicative Inverse of Complex Numbers We can think of division as multiplication by the inverse of a complex number. Given $z = a + b\imath$, the multiplicative inverse of z is given by:

$$\frac{1}{z} \; = \; \frac{\overline{z}}{|z|^2}$$

The Mandelbrot Set

The Mandelbrot set is the set of complex numbers c such that the sequence defined in equation (1) does not diverge.

$$z_{i+1} \; = \; z_i^2 \, + \, c \qquad \text{, where } \; i \, \in \, \{\, 0, 1, 2, 3, ... \,\} \quad \text{and } z_0 \; = \; 0 \tag{1}$$

Quoting from Wikipedia:

> Images of the Mandelbrot set exhibit an elaborate and infinitely complicated boundary that reveals progressively ever-finer recursive detail at increasing magnifications. The "style" of this repeating detail depends on the region of the set being examined. The set's boundary also incorporates smaller versions of the main shape, so the fractal property of self-similarity applies to the entire set, and not just to its parts.

Simple analysis shows that the sequence defined in equation (1) always diverges if $|c| > 2$. Observe that:

$$\begin{aligned} z_0 &= 0 \\ z_1 &= c \\ z_2 &= c^2 + c \\ &\vdots \end{aligned}$$

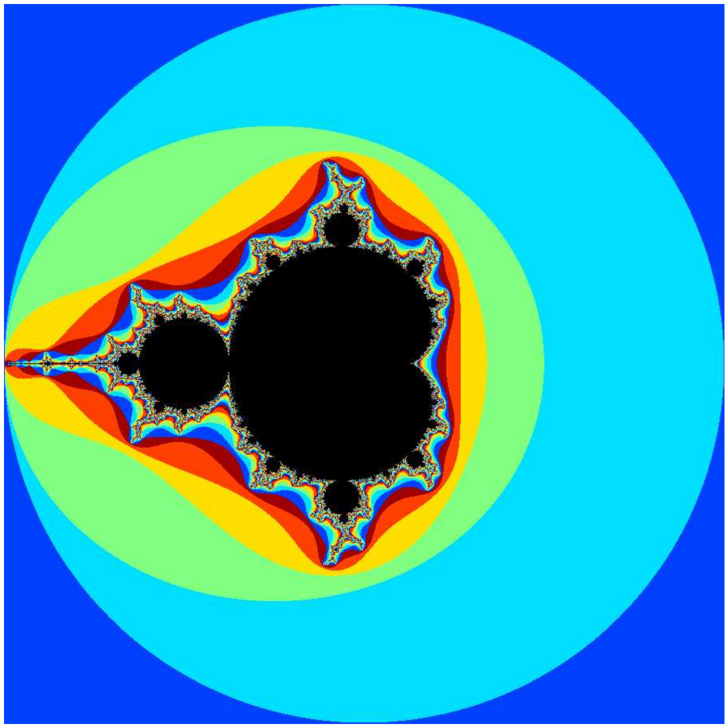
If $|c| > 2$, then $|z_1| > 2$. Further, whenever $|c| > 2$ it can be shown that $|z_{i+1}| > |z_i|$. Therefore, the sequence diverges whenever $|c| > 2$.

We can also observe that if there exists an n such that $|z_n| > 2$, then the sequence will diverge. I.e., for all values of $i > n$, $|z_i|$ can only continue to become larger.

These observations motivate the following definition:

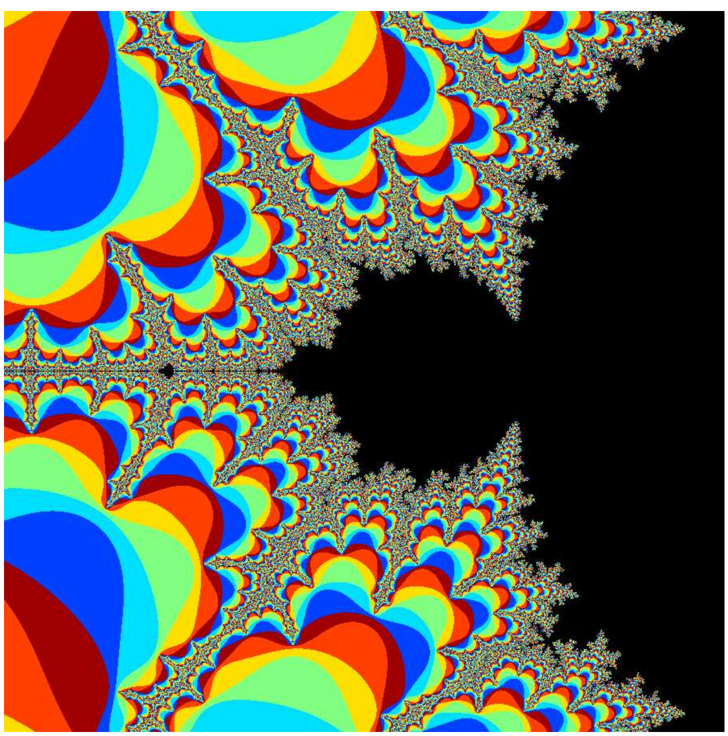
The **escape time** is the smallest n such that $|z_n| > 2$, if such an n exists. Notice that since $z_0 = 0$, the minimum escape time is 1. For some values of c, the sequence never escapes. For such values of c we assign an "artificial" escape value of zero.

We can produce a good visualization of the Mandelbrot set by defining a square matrix M where a row/column position within the array represents a point $a + b\imath$ in the complex plane where $-2 \leq a \leq 2$ and $-2 \leq b \leq 2$. The matrix entries are the escape times. Using use a different color for different escape times we see the following visualization.



In the visualization above, the range of values for c are taken from the rectangle bounded by -2 to 2 in both real and imaginary components.

Let us take a closer look. In this example, the values for c are chosen in a small rectangle. I.e., we let $c = a + b\imath$ where $-1.440 \leq a \leq -1.54$ and $-0.05 \leq b \leq 0.05$. Notice that the visualized rectangle is only 0.1 units on each side. Also, notice the self-similarity in the structure.



Mapping the 2-D Array to the Complex Plane

Our visualization of the Mandelbrot set requires us to map row-column positions in a square array to the complex plane. Let i denote the row index and let j denote the column index. If our square array is size $N \times N$, then the index values range as: $0 \leq i \leq N-1$ and $0 \leq j \leq N-1$. Suppose our corresponding square region in the complex plane is characterized by the lower left corner of the region of interest, and the length of one of the equal sides. Let $[a_{\min}, b_{\min}]$ denote the lower-left corner of the region of interest in the complex plane and let L denote the length of a side. Then the (i, j) position in the matrix is mapped to the complex number $a + b\imath$ using the formulas:

$$
\begin{aligned}
a &= \tfrac{i}{N-1} L + a_{\min} \\
b &= \tfrac{j}{N-1} L + b_{\min}
\end{aligned}
$$

To visualize the entire Mandelbrot set (as shown in the first image) use:

$$
\begin{aligned}
a_{\min} &= -2 \\
b_{\min} &= -2 \\
L &= 4
\end{aligned}
$$

In our definitions above, the row index i corresponds to the real axis and the column index j corresponds to the imaginary axis. The data visualization class `Img` that we are using transposes the image so that the horizontal direction in the displayed image corresponds to the real axis and the vertical direction in the displayed image corresponds to the imaginary axis.

Deciding When the Sequence Does Not Diverge

For some starting values in the complex plane, the iteration defined in equation (1) does not diverge. A simple way to detect non-divergence is to limit the number of iterations. If divergence is not detected within a pre-defined limit, we presume that divergence will never occur. A recommended limit is 1000 iterations.

In theory, there are points in the complex plane which will take more than 1000 iterations to diverge. In practice, the resolution of our display screens and the numerical resolution of our double precision numbers ensure that our chosen iteration limit (1000) will not affect the quality of our visualization.

Periodic Colormap for the Img Class